

2019 学年第一学期五校联考参考答案

高三年级数学学科

命题：杭州高级中学

一、选择题：

1-5 BCDAA 6-10 ADCDB

二、填空题：

11. $-2, 3$ 12. $\frac{1}{2}, \frac{4}{5}$ 13. $\{x|x < 0\}, \left\{x|\frac{1}{3} < x < \frac{5}{12}\right\}$

14. $-\frac{1}{16}, -\frac{1}{256}$ 15. 5 16. $\frac{2\sqrt{21}}{3}$ 17. 2

三、解答题：

18. 解：(I) $f(x) = \sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3} - \sqrt{3} \cos x$

$$= \frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x = \sin\left(x - \frac{\pi}{3}\right) \dots\dots\dots (4 \text{ 分})$$

当 $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ 时, $x - \frac{\pi}{3} \in \left[-\frac{5\pi}{6}, \frac{\pi}{6}\right]$, $\therefore f(x)$ 的值域是 $\left[-1, \frac{1}{2}\right]$ (3 分)

(II) $f(A) = \sin\left(A - \frac{\pi}{3}\right) = \frac{1}{3}$, 由于 $A - \frac{\pi}{3} \in \left(-\frac{\pi}{3}, \frac{2\pi}{3}\right)$, 则 $\cos\left(A - \frac{\pi}{3}\right) = \frac{2\sqrt{2}}{3}$

于是 $\sin A = \sin\left[\left(A - \frac{\pi}{3}\right) + \frac{\pi}{3}\right] = \frac{1+2\sqrt{6}}{6}$, (4 分)

由正弦定理 $\frac{a}{\sin A} = \frac{b}{\sin B}$ 得:

$$\sin B = \frac{b \sin A}{a} = \frac{2 \times \frac{1+2\sqrt{6}}{6}}{\sqrt{2}} = \frac{\sqrt{2}+4\sqrt{3}}{6} \dots\dots\dots (3 \text{ 分})$$

19. 解：(I) 证明：取 PA 的中点 O

则 $PA \perp DO$ -----①

$$\because OM \parallel AB, DC \parallel AB$$

$$\therefore OM \parallel DC$$

$\therefore O, D, C, M$ 四点共面

又 $\because AB // OM$ 且 $AB \perp PA$

$\therefore PA \perp OM$ -----②

由①②及 $DO \cap OM = O$

$\therefore PA \perp$ 面 $ODCM$

$\therefore PA \perp CM$ (5 分)

(II) 过点 B 作 OM 延长线的垂线且交 OM 延长线于 Q 点, 则 $BQ \perp OQ$

由 (I) 知 $\therefore PA \perp$ 面 $ODCM$, \therefore 面 $ODCM \perp$ 面 PAB

又 \because 面 $ODCM \cap$ 面 $PAB = OQ$, $\therefore BQ \perp$ 面 $ODCM$

$\therefore \angle BCQ$ 为求直线 BC 与平面 CDM 所成角

设 $AB = PA = DA = PD = \frac{1}{2}DC = 2$, 则 $BC = 2\sqrt{2}$ $BQ = 1$

$\therefore \sin \angle BCQ = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ (10 分)

20. 解: (1) 即 $a_n = 3^{n-1}$, $b_n = 2n - 1$, (3 分)

$$c_n = \frac{n \cdot (2n - 1)}{3^{n-1}},$$

$$\because c_{n+1} - c_n = \frac{(n+1)(2n+1)}{3^n} - \frac{n(2n-1)}{3^{n-1}} = \frac{-4n^2 + 6n + 1}{3^n}$$

令 $c_{n+1} - c_n > 0$ 即 $4n^2 - 6n - 1 < 0$ 解得 $n = 1 \therefore c_2 > c_1$

当 $n \geq 2$ 时, $c_{n+1} - c_n < 0$, 此时数列 $\{c_n\}$ 单调递减

\therefore 数列 $\{c_n\}$ 中的最大项为第 2 项, $\therefore k = 2$ (5 分)

(II) $T_n = 1 + 3 \cdot 3 + 3^2 \cdot 5 + \cdots + 3^{n-2} \cdot (2n - 3) + 3^{n-1} \cdot (2n - 1)$

$$3T_n = 3 \cdot 1 + 3^2 \cdot 3 + 3^3 \cdot 5 + \cdots + 3^{n-1} \cdot (2n - 3) + 3^n \cdot (2n - 1)$$

$$\text{相减得: } -2T_n = 1 + 2 \cdot \frac{3(1 - 3^{n-1})}{1 - 3} - 3^n \cdot (2n - 1)$$

于是: $T_n = 3^n(n - 1) + 1$ (7 分)

解: (1) 左焦点 F 的坐标为 $(-1, 0)$

$$y = k_1(x+1) \text{ 代入 } \frac{x^2}{2} + y^2 = 1$$

$$(1+2k_1^2)x^2 + 4k_1^2x + 2k_1^2 - 2 = 0$$

$$\text{设 } A(x_1, y_1), B(x_2, y_2), M(x_0, y_0)$$

$$\text{则 } x_1 + x_2 = -\frac{4k_1^2}{1+2k_1^2}, x_1x_2 = \frac{2k_1^2-2}{1+2k_1^2}$$

$$x_0 = \frac{x_1 + x_2}{2} = -\frac{2k_1^2}{1+2k_1^2}, y_0 = k_1(x_0+1) = \frac{k_1}{1+2k_1^2}$$

$$k_2 = k_{OM} = \frac{1}{-2k_1}, \text{ 所以 } k_1k_2 = -\frac{1}{2}$$

$$(2) |AB| = \sqrt{1+k_1^2} |x_1 - x_2| = \sqrt{1+k_1^2} \frac{2\sqrt{2}\sqrt{1+k_1^2}}{1+2k_1^2} = \frac{2\sqrt{2}(1+k_1^2)}{1+2k_1^2},$$

$$y = k_2x \text{ 代入 } \frac{x^2}{2} + y^2 = 1, \text{ 得 } x_D = -\frac{\sqrt{2}}{\sqrt{1+2k_2^2}}, x_C = \frac{\sqrt{2}}{\sqrt{1+2k_2^2}}$$

$$|MC| \cdot |MD| = \sqrt{1+k_2^2} \left| x_0 - \frac{\sqrt{2}}{\sqrt{1+2k_2^2}} \right| \cdot \sqrt{1+k_2^2} \left| x_0 + \frac{\sqrt{2}}{\sqrt{1+2k_2^2}} \right|$$

$$= (1+k_2^2) \left| x_0^2 - \frac{2}{1+2k_2^2} \right| = (1+k_2^2) \left| \left(\frac{2k_1^2}{1+2k_1^2} \right)^2 - \frac{2}{1+2k_2^2} \right|$$

$$\text{因为 } |MB|^2 = |MC| \cdot |MD|, \text{ 所以 } \frac{1}{4}|AB|^2 = |MC| \cdot |MD|,$$

$$\frac{2(1+k_1^2)^2}{(1+2k_1^2)^2} = \left| \left(\frac{2k_1^2}{1+2k_1^2} \right)^2 - \frac{4k_1^2}{1+2k_1^2} \right|, \text{ 解得 } k_1^2 = \frac{1}{2}$$

$$\text{所以 } \{k_1, k_2\} = \left\{ \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\}, \text{ 由对称性, 不妨设 } k_1 = \frac{\sqrt{2}}{2}, k_2 = -\frac{\sqrt{2}}{2}$$

$$\text{直线 } CD \text{ 方程 } \sqrt{2}x + 2y = 0, \text{ 点 } F \text{ 到直线 } CD \text{ 距离分别是 } d_F = \frac{\sqrt{3}}{3}$$

$$|CD| = \sqrt{1+k_2^2} |x_C - x_D| = \sqrt{1+k_2^2} \cdot \frac{2\sqrt{2}}{\sqrt{1+2k_2^2}} = \sqrt{6}$$

$$\text{四边形 } FCBD \text{ 的面积为 } \frac{1}{2}|CD| \cdot d_F = \frac{\sqrt{2}}{2}$$

$$22. (1) \text{ 当 } a = -1 \text{ 时, } f(x) = e^x - \sqrt{x+1}, x \geq -1$$

$$f'(x) = e^x - \frac{1}{2\sqrt{x+1}}, \text{ 显然, } f'(x) \text{ 在 } (-1, +\infty) \text{ 上递增,}$$

$$\text{又 } f'(-\frac{1}{2}) = \frac{1}{\sqrt{e}} - \frac{1}{\sqrt{2}} < 0, f'(0) = 1 - \frac{1}{2} > 0$$

所以 $f'(x) = e^x - \frac{1}{2\sqrt{x+1}} = 0$ 在 $\left(-\frac{1}{2}, 0\right)$ 有唯一零点

所以 $-\frac{1}{2} < x_0 < 0 \dots\dots\dots (6 \text{ 分})$

(2) (i) 证明: 设 $h(x) = f(x) - (1+x + \frac{1}{2}x^2 + a\sqrt{x+1}) = e^x - (1+x + \frac{1}{2}x^2), x \geq 0$

则 $h'(x) = e^x - (1+x), x \geq 0$

那么 $h''(x) = e^x - 1, x \geq 0$

当 $x > 0$ 时, $f'''(x) = e^x - 1 > 0$

所以 $f'(x) = e^x - (1+x)$ 在 $(0, +\infty)$ 上递增

故 $f'(x) \geq f'(0) = 0$

所以 $f(x) = e^x - (1+x + \frac{1}{2}x^2)$ 在 $(0, +\infty)$ 上递增

故 $f(x) \geq f(0) = 0$

所以 $e^x \geq 1+x + \frac{1}{2}x^2 \dots\dots\dots (4 \text{ 分})$

(ii) 在 $2 + \frac{5}{4}x + \frac{a}{2}x^2 \leq \frac{e^x + a\sqrt{x+1}}{a}$ 中, 令 $x=0$, 得 $0 < a \leq 1$

当 $0 < a \leq 1$ 时,

$$\frac{e^x + a\sqrt{x+1}}{a} - (2 + \frac{5}{4}x + \frac{a}{2}x^2) = \frac{e^x}{a} + \sqrt{x+1} - (2 + \frac{5}{4}x + \frac{a}{2}x^2)$$

$$\geq \frac{e^x}{1} + \sqrt{x+1} - (2 + \frac{5}{4}x + \frac{1}{2}x^2)$$

设 $g(x) = e^x + \sqrt{x+1} - (2 + \frac{5}{4}x + \frac{1}{2}x^2)$, 则 $g'(x) = e^x + \frac{1}{2\sqrt{x+1}} - (\frac{5}{4} + x)$

由 (i) 得, 当 $x \geq 0$ 时

$$g'(x) = e^x + \frac{1}{2\sqrt{x+1}} - (\frac{5}{4} + x) \geq 1+x + \frac{1}{2}x^2 + \frac{1}{2\sqrt{x+1}} - (\frac{5}{4} + x)$$

$$= \frac{1}{2}x^2 + \frac{1}{2\sqrt{x+1}} - \frac{1}{4}, \text{ 当 } x \geq 1 \text{ 时, } \frac{1}{2}x^2 + \frac{1}{2\sqrt{x+1}} - \frac{1}{4} > \frac{1}{2}x^2 - \frac{1}{4} \geq \frac{1}{2} - \frac{1}{4} > 0$$

当 $0 \leq x < 1$ 时, $\frac{1}{2}x^2 + \frac{1}{2\sqrt{x+1}} - \frac{1}{4} \geq \frac{1}{2\sqrt{x+1}} - \frac{1}{4} > \frac{1}{2\sqrt{2}} - \frac{1}{4} = 0$

所以当 $x \geq 0$ 时, $g'(x) > 0$, $g(x) = e^x + \sqrt{x+1} - (2 + \frac{5}{4}x + \frac{1}{2}x^2)$ 在 $(0, +\infty)$ 上递增

所以 $g(x) \geq g(0) = 0$, 因此当 $0 < a \leq 1$ 时, 不等式 $2 + \frac{5}{4}x + \frac{a}{2}x^2 \leq \frac{f(x)}{a}$ 对任意 $x \geq 0$ 恒成立。